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# [0,1] Truncated Exponential Marshall-Olkin-Gompertz Distribution: Properties and Applications

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**Abstract:**

In this paper, a new compound distribution termed as [0,1]Truncated Exponential Marshall-Olkin-Gompertz (TEMO-Go)is introduced. Several essential statistical properties of TEMO-Go distribution were studied. The estimation of distribution parameters was performed using the maximum likelihood estimation method, the flexibility of the new distribution was examined using two real data. The suitability of the proposed distribution indicates that it performs positively when compared to the existing distributions.

**Keywords**: New distribution, Moment, MLE, Gompertz distribution.

# [0, 1]توزيع الأسي مارشال أولكين جومبيرتز المبتور: الخصائص مع التطبيقات

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**المستخلص:**

في هذا البحث، تم تقديم توزيع مركب جديد يسمى:
[0,1] Truncated Exponential Marshall-Olkin-Gompertz (TEMO-Go)
تمت دراسة العديد من الخصائص الإحصائية الأساسية لتوزيع TEMO-Go. تم إجراء تقدير معلمات التوزيع باستخدام طريقة تقدير الامكان الاعظم، وتم فحص مرونة التوزيع الجديد باستخدام اثنين من البيانات الحقيقية. تشير ملاءمة التوزيع المقترح إلى أنه يعمل بشكل إيجابي عند مقارنته بالتوزيعات الحالية.

**الكلمات المفتاحية:** توزيع جديد، عزم، تقدير الامكان الاعظم، توزيع جومبيرتز.

## 1. Introduction:

The Gompertz distribution is both skewed to the right and to the left. It is a generalization of the exponential distribution and is commonly used in many applied problems. The Gompertz distribution has been applied in computer (Ohishi et al., 2009).It received interest from demographers and its ability to model for growth, in particular in human epidemiological and biomedical studies (Lenart, 2012).

Recently, there has been an extraordinary enthusiasm for providing more flexible distributions by extending the range of classical distributions by adding one or more shape parameters to the basic distribution PDF. (Marshall, and Olkin, 1997) a new family by adding a shape parameter to the primary distribution, many researchers used it to find and study new distributions. (Nwezza et al., 2020) using the Marshall Olkin family by expanding the Gumbel-Lomax distribution and applying it to aircraft windshield failure data, in the same year (Khaleel et al., 2020) and (Al-Noor, and Hadi, 2020) using the Marshall Olkin family to produce new distributions.

(Alizadeh et al., 2017) using the Gompertz distribution to generate a new family that has high flexibility in describing different data, (Oguntunde et al., 2019) using the Gompertz family, a new distribution called Gompertz Fréchet was found and applied to Engineering data, (Eghwerido et al., 2020) using the Gompertz family by finding a new distribution called Gompertz-Alpha Power Inverted Exponential, also in the same year (Khaleel et al., 2020) using the Gompertz family by extending the Flexible Weibull distribution , and the application of these distributions to different types of data.

There is still a lot of space left to develop suitable new estimated distributions under various conditions, so based on the Marshall-Olkin family, this paper presented a proposed new distribution called[0,1] Truncated Exponential Marshall-Olkin-Gompertz (TEMO-Go).

The research aims to study some mathematical properties of the proposed TEMO-Go distribution and prove that the proposed distribution is more flexible than other distributions of real data.

This paper is organized as follows: In the second section, the cumulative function, the probability density function, the survival function, the risk function, and the cumulative risk function of the TEMO-Go distribution are defined. In the third section, the probability density function and the cumulative function are expanded. The fourth section the mathematical properties, including moments, moment generating function, quantity function, ordered statistics, and entropy are studied. Section five contained the maximum likelihood method for estimating the model parameters. and in the sixth section the application. The research concluded with conclusions and recommendations.

**2. [0,1] Truncated Exponential Marshall-Olkin-Gompertz (TEMO-Go):**

According to (Atanda et al., 2020), The cumulative distribution function CDF and probability density function PDF of the Gompertz distribution with parameters β and λ are given as:

$$G(x;\beta,\lambda) = 1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)} \tag{1}$$

And

$$g(x;\beta,\lambda) = \beta e^{\lambda x} e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)} \tag{2}$$

Where $x > 0\,, \beta, \lambda > 0$

We follow the method by (Abid et al., 2018), and the family introduced by (Al-Noor, and Hadi, 2020) with the CDF and PDF given as:

$$F(x;\theta,\alpha) = \frac{1-e^{-\theta\left(\frac{G(x;\xi)}{\alpha+\bar{\alpha}G(x;\xi)}\right)}}{1-e^{-\theta}} \tag{3}$$

ξ are the parameters of the main distribution .

$$f(x;\theta,\alpha) = \frac{\theta\alpha g(x) e^{-\theta\left(\frac{G(x;\xi)}{\alpha+\bar{\alpha}G(x;\xi)}\right)}}{(1-e^{-\theta})(\alpha+\bar{\alpha}G(x;\xi))^2} \tag{4}$$

where $x > 0, \theta, \alpha > 0$ are related to the shape parameters.

Substituting equation (1) into (3) above and simplifying, we obtain CDF for the new distribution TEMO-Go of a random variable X as follows:

$$F(x;\theta,\alpha,\beta,\lambda) = \frac{1-e^{-\theta\left(\frac{1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}}{\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]}\right)}}{1-e^{-\theta}} \tag{5}$$

Substituting equation (1) into (4) above, we obtain PDF for the new distribution TEMO-Go:

$$f(x;\theta,\alpha,\beta,\lambda) = \frac{\theta\alpha\beta e^{\lambda x} e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)} e^{-\theta\left(\frac{1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}}{\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]}\right)}}{(1-e^{-\theta})\left(\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]\right)^2} \tag{6}$$

The hazard function of the TEMO-Go:

$$h(x;\theta,\alpha,\beta,\lambda)=\frac{f(x;\theta,\alpha,\beta,\lambda)}{1-F(x;\theta,\alpha,\beta,\lambda)}=\frac{\dfrac{\theta\alpha\beta e^{\lambda x}e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}e^{-\theta\left(\frac{1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}}{\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]}\right)}}{\left(1-e^{-\theta}\right)\left(\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]\right)^{2}}}{1-\left\{\dfrac{1-e^{-\theta\left(\frac{1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}}{\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]}\right)}}{1-e^{-\theta}}\right\}} \tag{7}$$

where $x>0,\theta,\alpha,\beta,\lambda>0$ are related to the shape parameters.

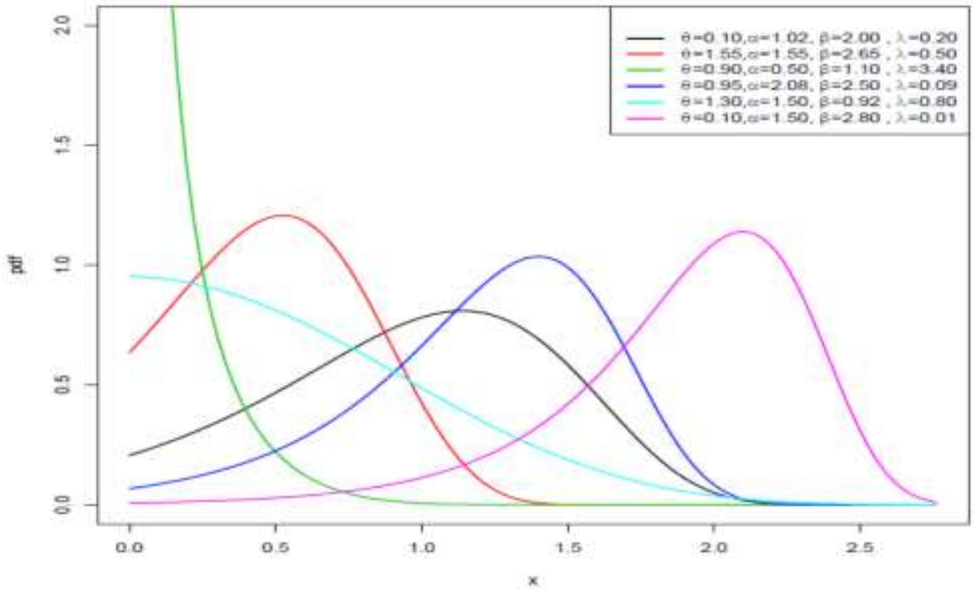


Figure (1): Plot of PDF of the TEMO-Go distribution with different values of parameters using R software

**3. Expansion of functions:**

In this section, the PDF and the CDF for the TEMO-Go distribution will be expanded for the purpose of studying some of the statistical characteristics and characteristics of the new distribution. By taking Equation No. (4) and simplifying it:

Exponential expansion and binomial series expansion multiple times respectively as (Khaleel et al., 2020) and (Al-Noor, and Hadi, 2020):

$e^{-q}=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}q^m$ , $(1-w)^{-b}=\sum_{k=0}^{\infty}\frac{\Gamma(b+k)}{k!\Gamma(b)}w^k$, $(1-w)^b=\sum_{j=0}^{\infty}(-1)^j\binom{b}{j}w^j$

Where $\Gamma(b)=\int_0^{\infty}t^{b-1}e^{-t}dt$

Now we simplify equation (4) as follows:

$$f(x;\theta,\alpha)=\frac{\theta\alpha g(x;\xi)}{(1-e^{-\theta})}e^{-\theta\left(\frac{G(x;\xi)}{\alpha+\bar{\alpha}G(x;\xi)}\right)}\left(\alpha+\bar{\alpha}G(x;\xi)\right)^{-2} \tag{8}$$

By using Exponential expansion we get:

$$e^{-\theta\left(\frac{G(x;\xi)}{\alpha+\bar{\alpha}G(x;\xi)}\right)}=\sum_{m=0}^{\infty}\frac{(-1)^m\theta^m}{m!}\left(\frac{G(x;\xi)}{\alpha+\bar{\alpha}G(x;\xi)}\right)^m$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m\theta^m}{m!}\left(G(x;\xi)\right)^m\left(\alpha+\bar{\alpha}G(x;\xi)\right)^{-m} \tag{9}$$

After some algebra we get:

$$f(x;\theta,\alpha)=\sum_{m=0}^{\infty}\frac{\theta^{m+1}\alpha(-1)^m}{(1-e^{-\theta})m!}g(x;\xi)\left(G(x;\xi)\right)^m\left(\alpha+\bar{\alpha}G(x;\xi)\right)^{-2-m} \tag{10}$$

By using binomial series expansion we get:

$$\left(\alpha+\bar{\alpha}G(x;\xi)\right)^{-(2+m)}=\left(1-\bar{\alpha}\bar{G}(x;\xi)\right)^{-(2+m)}=\sum_{k=0}^{\infty}\frac{\Gamma(2+m+k)}{k!\Gamma(2+m)}(\bar{\alpha})^k\left(\bar{G}(x;\xi)\right)^k \tag{11}$$

Substituting (11) into (10) we get:

$$f(x;\theta,\alpha)=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{\theta^{m+1}\alpha(-1)^m(\bar{\alpha})^k}{(1-e^{-\theta})\Gamma(2+m)m!k!}g(x;\xi)\left(G(x;\xi)\right)^m\left(1-G(x;\xi)\right)^k \tag{12}$$

Again ,by using binomial series expansion we get:

$$(1-G(x;\xi))^k=\sum_{j=0}^{\infty}(-1)^j\binom{k}{j}\left(G(x;\xi)\right)^j \tag{13}$$

Then:

$$f(x;\theta,\alpha)=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\theta^{m+1}\alpha(-1)^{m+j}(\bar{\alpha})^k}{(1-e^{-\theta})\Gamma(2+m)m!k!}\binom{k}{j}g(x;\xi)\left(G(x;\xi)\right)^{m+j} \tag{14}$$

From substituting equations (1) and (2) into equation (14) we get

$$f(x;\theta,\alpha,\beta,\lambda)=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\theta^{m+1}\beta\alpha(-1)^{m+j}(\bar{\alpha})^k}{(1-e^{-\theta})\Gamma(2+m)m!k!}\binom{k}{j}e^{\lambda x}e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\left(1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right)^{m+j} \tag{15}$$

Now we simplify and use the binomial series expansion again:

$$\left(1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right)^{m+j}=\sum_{i=0}^{\infty}(-1)^i\binom{m+j}{i}e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)i} \tag{16}$$

Then:

$$f(x;\theta,\alpha,\beta,\lambda)=\sum_{m=k=j=i=0}^{\infty}\frac{\theta^{m+1}\beta\alpha(-1)^{m+j+i}(\bar{\alpha})^k}{(1-e^{-\theta})\Gamma(2+m)m!k!}\binom{k}{j}\binom{m+j}{i}e^{\lambda x}e^{\frac{\beta}{\lambda}(i+1)}e^{-\frac{\beta}{\lambda}(i+1)e^{\lambda x}} \tag{17}$$

Again, using series expansion of $e^{-\frac{\beta}{\lambda}(i+1)e^{\lambda x}}$, we get (Atanda et al., 2020):

$$f(x;\theta,\alpha,\beta,\lambda)=\sum_{m=k=j=i=r=0}^{\infty}T_{m,k,j,i,r}e^{\lambda(r+1)x} \tag{18}$$

Where $T_{m,k,j,i,r}=\dfrac{\theta^{m+1}\beta\alpha(-1)^{m+j+i+r}(\bar{\alpha})^k\left(\beta(i+1)\right)^r}{(1-e^{-\theta})\Gamma(2+m)m!k!\lambda^r}\binom{k}{j}\binom{m+j}{i}e^{\frac{\beta}{\lambda}(i+1)}$

Equation (18) can be used in finding many mathematical properties as we will explain that.

## 4. Statistical properties of TEMO-Go distribution

In this section , We present the most important statistical properties of the TEMO-Go distribution.

### 4.1. Moments:

We can obtain the n-degree moment of the TEMO-Go distribution from the following relationship by relying on the probability density function expanded by equation (18):

$$\acute{\mu}_n = E(X^n) = \int_0^{\infty} x^n f(x;\theta,\alpha,\beta,\lambda)dx = \sum_{m=k=j=i=r=0}^{\infty} T_{m,k,j,i,r} \int_0^{\infty} x^n e^{\lambda(r+1)x}\,dx$$

Where $T_{m,k,j,i,r} = \frac{\theta^{m+1}\beta\alpha(-1)^{m+j+i+r}(\bar{\alpha})^k(\beta(i+1))^r}{(1-e^{-\theta})\Gamma(2+m)\,m!k!\lambda^r}\binom{k}{j}\binom{m+j}{i}e^{\frac{\beta}{\lambda}(i+1)}$

let $-y = \lambda(r+1)x \Rightarrow x = -y[\lambda(r+1)]^{-1}, -\frac{dy}{dx}=\lambda(r+1) \Rightarrow dx = -\frac{dy}{\lambda(r+1)}$

$$\acute{\mu}_n = \sum_{m=k=j=i=r=0}^{\infty} T_{m,k,j,i,r}\left(-\frac{1}{\lambda(r+1)}\right)^{n+1} \int_0^{\infty} y^n e^{-y}dy$$

By finding the integral, we obtain the final momentum formula for the new TEMO-Go distribution as follows:

$$\acute{\mu}_n = E(X^n) = \sum_{m=k=j=i=r=0}^{\infty} T_{m,k,j,i,r}\frac{(-1)^{n+1}\Gamma(n+1)}{[\lambda(r+1)]^{n+1}} \qquad (19)$$

Where $T_{m,k,j,i,r} = \frac{\theta^{m+1}\beta\alpha(-1)^{m+j+i+r}(\bar{\alpha})^k(\beta(i+1))^r}{(1-e^{-\theta})\Gamma(2+m)\,m!k!\lambda^r}\binom{k}{j}\binom{m+j}{i}e^{\frac{\beta}{\lambda}(i+1)}$

### 4.2. Moment Generating Function:

The moment generating function M.G.F of the continuous variable X of the TEMO-Go distribution is as follows:

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x;\theta,\alpha,\beta,\lambda)dx$$

Using equation (19) and expanding the exponential function, we can find the M.G.F function for the TEMO-Go distribution where according to Taylor series we have:

$$E(e^{tx}) = \sum_{l=0}^{\infty}\frac{t^l}{l!}E(x^l)\,,\, M_x(t) = \sum_{l=0}^{\infty}\frac{t^l}{l!}\int_0^{\infty} x^l f(x)dx = \sum_{l=0}^{\infty}\frac{t^l}{l!}E(x^l) = \sum_{l=0}^{\infty}\frac{t^l}{l!}[\acute{\mu}_n]$$

$$M_x(t) = \sum_{m=k=j=i=r=l=0}^{\infty} \mathcal{V}_{m,k,j,i,r,l}\frac{(-1)^{n+1}\Gamma(n+1)}{[\lambda(r+1)]^{n+1}} \qquad (20)$$

Where $\mathcal{V}_{m,k,j,i,r,l} = \frac{\theta^{m+1}\beta\alpha(-1)^{m+j+i+r}(\bar{\alpha})^k(\beta(i+1))^r t^l}{(1-e^{-\theta})\Gamma(2+m)\,m!k!\lambda^r l!}\binom{m+j}{i}e^{\frac{\beta}{\lambda}(i+1)}$

### 4.3. Quantile function:

According to (Hyndman, and Fan, 1996) the Quantile function of any distribution is defined by the formula $Q(u) = F^{-1}(x)$ where Q (u) is the quantity function of F (x) for every $0 < u < 1$ To obtain the Quantile function of the TEMO-Go distribution, we invert the CDF given in Equation (5). Note that:

$$\frac{1 - e^{-\theta\left(\frac{1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}}{\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]}\right)}}{1 - e^{-\theta}} = u$$

Using some algebraic operations, we get the formula:

$$Q(u) = \frac{1}{\lambda} + \frac{1}{\lambda} ln\left\{\frac{-\lambda}{\beta} ln\left[1 - \left[\frac{-\alpha\, ln[1 - u(1 - e^{-\theta})]}{\theta + \bar{\alpha}\, ln[1 - u(1 - e^{-\theta})]}\right]\right]\right\} \quad (21)$$

**4.4. Order statistics:**

Let $X_1, X_2, \dots, X_n$ be a random sample of size n, from the TEMO-Go distribution, which has a cumulative function in Equation No. (5) and the probability density function in Equation No. (6), so the probability density function PDF for the ordered statistics we write it as follows:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{s=0}^{n-i} (-1)^s \binom{n-i}{s} f_{TEMO-Go}(x)[F_{TEMO-Go}(x)]^{i+s-1} \quad (22)$$

Substituting (5) and (6) into equation (22) we get:

$$f_{i:n}(x) = K \sum_{s=0}^{n-i} (-1)^s \binom{n-i}{s} \left[\frac{\theta\alpha\beta e^{\lambda x} e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)} e^{-\theta\left(\frac{1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}}{\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]}\right)}}{(1-e^{-\theta})\left(\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]\right)^2}\right]\left[\frac{1-e^{-\theta\left(\frac{1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}}{\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]}\right)}}{1-e^{-\theta}}\right]^{i+s-1} \quad (23)$$

Where K= n!/(i-1)!(n-i)!

If we substitute i = 1, we get its lowest ranked statistic:

$$f_{1:n}(x) = n \sum_{s=0}^{n-1} (-1)^s \binom{n-1}{s} \left[\frac{\theta\alpha\beta e^{\lambda x} e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)} e^{-\theta\left(\frac{1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}}{\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]}\right)}}{(1-e^{-\theta})\left(\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]\right)^2}\right]\left[\frac{1-e^{-\theta\left(\frac{1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}}{\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]}\right)}}{1-e^{-\theta}}\right]^{s} \quad (24)$$

If we substitute i = n, we get its largest statistic in order:

$$f_{n:n}(x) = n \left[\frac{\theta\alpha\beta e^{\lambda x} e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)} e^{-\theta\left(\frac{1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}}{\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]}\right)}}{(1-e^{-\theta})\left(\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]\right)^2}\right]\left[\frac{1-e^{-\theta\left(\frac{1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}}{\alpha+\bar{\alpha}\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]}\right)}}{1-e^{-\theta}}\right]^{n+s-1} \quad (25)$$

### 4.5. Entropy

The entropy of a random variable is a measure of uncertainty change and it has been used in various situations in science and engineering. Rényi Entropy is defined by:

$$I_\alpha(x) = \frac{1}{1-\alpha} log \int_0^\infty f(x;\theta,\alpha,\beta,\lambda)^\alpha dx$$

Thus the Rényi entropy of the TEMO-Go distribution can be obtained.

$$I_\alpha(x) = \frac{1}{1-\alpha} \log \int_0^\infty \sum_{m=k=j=i=r=0}^{\infty} T_{m,k,j,i,r} [e^{\lambda(r+1)x}]^\alpha$$

$$I_\alpha(x) = \frac{1}{1-\alpha} log \sum_{m=k=j=i=r=0}^{\infty} T_{m,k,j,i,r} \int_0^\infty e^{\lambda\alpha(r+1)x} dx \tag{26}$$

Where $T_{m,k,j,i,r} = \frac{\theta^{m+1}\beta\alpha(-1)^{m+j+i+r}(\bar{\alpha})^k(\beta(i+1))^r}{(1-e^{-\theta})\Gamma(2+m)m!k!\lambda^r}\binom{m+j}{i}e^{\frac{\beta}{\lambda}(i+1)}$

By finding the integral as in equation (19), the final form of the Rényi Entropy of the TEMO-Go.

$$I_\alpha(x) = \frac{1}{1-\alpha} log \sum_{m=k=j=i=r=0}^{\infty} T_{m,k,j,i,r} \frac{(-1)^{n+1}\Gamma(n+1)}{[\lambda\alpha(r+1)]^{n+1}} \tag{27}$$

## 5. Maximum likelihood estimations:

Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n from the TEMO-Go distribution with unknown parameters, $\lambda$, $\beta$, $\alpha$ and $\theta$ Predefined.

The likelihood function is given by:

$$L(\underline{X}/\alpha,\beta,\theta,\lambda) = \frac{(\theta\beta\alpha)^n e^{\lambda\sum_{i=1}^n x_i} e^{-\frac{\beta}{\lambda}\sum_{i=1}^n (e^{\lambda x_i}-1)} e^{-\theta\sum_{i=1}^n\left(\frac{1-e^{-\frac{\beta}{\lambda}(e^{\lambda x_i}-1)}}{\alpha+(1-\alpha)\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x_i}-1)}\right]}\right)}}{(1-e^{-\theta})^n \prod_{i=1}^n\left(\alpha+(1-\alpha)\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x_i}-1)}\right]\right)^2} \tag{28}$$

Let the log-likelihood function be $l = logL(\underline{X}/\alpha,\beta,\theta,\lambda)$ therefore.

$$\begin{aligned} l = nlog\theta + nlog\alpha + nlog\beta - nlog(1-e^{-\theta}) + \lambda\sum_{i=1}^n x_i - \frac{\beta}{\lambda}\sum_{i=1}^n (e^{\lambda x_i}-1) \\ - \theta\sum_{i=1}^n\left(\frac{1-e^{-\frac{\beta}{\lambda}(e^{\lambda x_i}-1)}}{\alpha+(1-\alpha)\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x_i}-1)}\right]}\right) \\ - 2\sum_{i=1}^n log\left(\alpha+(1-\alpha)\left[1-e^{-\frac{\beta}{\lambda}(e^{\lambda x_i}-1)}\right]\right) \end{aligned} \tag{29}$$

We take the partial derivative of the log likelihood function with respect to parameters $\alpha$, $\beta$, $\theta$ and $\lambda$ in order to obtain an estimate of the probability of each parameter.

$$\frac{\partial(l)}{\partial\alpha}=\frac{n}{\alpha}-\theta\sum_{i=1}^{n}\left(\frac{-\left(1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}\right)\left(e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}\right)}{\left(\alpha+(1-\alpha)\left[1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}\right]\right)^2}\right)-2\sum_{i=1}^{n}\frac{e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}}{\left(\alpha+(1-\alpha)\left[1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}\right]\right)} \qquad (30)$$

$$\frac{\partial(l)}{\partial\beta}=\frac{n}{\beta}-\frac{1}{\lambda}\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right) \qquad (31)$$

$$-\theta\sum_{i=1}^{n}\left(\frac{\alpha+(1-\alpha)\left[1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}\right]\left(\frac{\left(e^{\lambda x_i}-1\right)}{\lambda}e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}\right)-\left(1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}\right)\left(\frac{(-\alpha+1)\left(e^{\lambda x_i}-1\right)}{\lambda}e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}\right)}{\left(\alpha+(1-\alpha)\left[1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}\right]\right)^2}\right)$$

$$-2\sum_{i=1}^{n}\frac{\frac{(-\alpha+1)\left(e^{\lambda x_i}-1\right)}{\lambda}e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}}{\left(\alpha+(1-\alpha)\left[1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}\right]\right)}$$

$$\frac{\partial(l)}{\partial\theta}=\frac{n}{\theta}+\frac{ne^{-\theta}}{\left(1-e^{-\theta}\right)}-\sum_{i=1}^{n}\left(\frac{1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}}{\alpha+(1-\alpha)\left[1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}\right]}\right) \qquad (32)$$

$$\frac{\partial(l)}{\partial\lambda}=\sum_{i=1}^{n}x_i-\beta\sum_{i=1}^{n}\left(\frac{x_i e^{\lambda x_i}-1}{\lambda}-\frac{e^{\lambda x_i}-1}{\lambda^2}\right) \qquad (33)$$

$$-\theta\sum_{i=1}^{n}\left(\frac{R\left(\beta e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}\left(\frac{x_i}{\lambda}e^{\lambda x_i}-\frac{e^{\lambda x_i}-1}{\lambda^2}\right)\right)-\left[D\left(\beta e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}(-\alpha+1)\left(\frac{x_i e^{\lambda x_i}}{\lambda}-\frac{e^{\lambda x_i}-1}{\lambda^2}\right)\right)\right]}{R^2}\right)$$

$$-2\sum_{i=1}^{\infty}\left(\frac{\left(\beta e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}(-\alpha+1)\left(\frac{x_i e^{\lambda x_i}}{\lambda}-\frac{e^{\lambda x_i}-1}{\lambda^2}\right)\right)}{\left(\alpha+(1-\alpha)\left[1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}\right]\right)}\right)$$

Where $D=1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}$ and $R=\alpha+(1-\alpha)\left[1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x_i}-1\right)}\right]$

We note that equations (30), (31), (32), (33) after being equal to zero are difficult to solve manually. Mathematical programs should be used in this study. We use the R program.

## 6. Applications:

This section presents two datasets, The section compares the performance of the Truncated Exponential Marshall-Olkin-Gompertz (TEMOGo) to that of Beta Gompertz distribution (BGo) (Jafari et al., 2014), Kumaraswamy Gompertz distribution (KuGo) (da Silva et al., 2015), Exponential Generalized Gompertz distribution (EGGo) (New), Weibull Gompertz distribution (WGo) (New), Gompertz Gompertz distribution (GoGo) (New), Gompertz distribution (Go) (Pollard, & Valkovics, 1992). Aiming to evaluate the performance of the models listed above, Statistical criteria such as the Negative Log Likelihood (NLL), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC). Respectively as in the following equations:

$AIC = -2l + 2K$ , $BIC = -2l + k\log(n)$, $CAIC = AIC + \frac{2k(k+1)}{n-k-1}$,$HQIC = 2k\ln[\ln(n)] - 2l$

Where value (K) Like the number of parameters of the distribution, value (n) Represent the sample size and ($l$) is the maximized value of the log-likelihood function under the considered model. The unknown parameters of the distributions were estimated using the greatest possibility method, and that

Using R language program to find the best distribution that matches the data.

**6.1. Dataset1:**

The dataset1 represent the strength data measured in GPA, for single carbon fibers were tested under tension at gauge lengths of 1, 10, 20 and 50 mm. The dataset1 has previously been extracted from (Ahmad, and Hussain, 2017). For illustrative purpose, we consider only the data set consisting the single fibers of 20 mm, with a sample of size 63.

The data are:

| |
|---|
| 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977,2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020 |

Table (1): Summary statistics for dataset 1

| Var | N | Mean | Sd | Median | Min | Max | Skew | Kurtosis |
|---|---|---|---|---|---|---|---|---|
| | 63 | 3.06 | 0.62 | 3 | 1.9 | 5.02 | 0.62 | 0.18 |

Note the value of skewness is positive, and this indicates that the data is skewed on the right. As for kurtosis, its value is positive, and this means that the data with thin flatness is close to the normal distribution, meaning that the mean and median of the data are close in value.

Table (2): an estimate of the parameters for the distributions with the statistical criteria
(NLL, AIC, CAIC, BIC, HQIC) for dataset 1

| **Model** | **Est para** | **-LL** | **AIC** | **CAIC** | **BIC** | **HQIC** |
|---|---|---|---|---|---|---|
| TEMOGo | 37.790<br>2.7995<br>0.1183<br>1.4048 | 56.3 | 120.7 | 121.4 | 129.2 | 124.0 |
| BGo | 1.5237<br>0.2858<br>1.3029<br>0.0684 | 65.2 | 138.4 | 139.1 | 147.8 | 141.8 |
| KuGo | 1.3934<br>0.2460<br>1.2232<br>0.0961 | 66.7 | 141.5 | 142.1 | 150.0 | 144.8 |
| EGGo | 0.9645<br>1.6318<br>1.1536<br>0.0361 | 65.2 | 138.5 | 139.2 | 147.1 | 141.9 |
| WGo | 1.6012<br>1.0532<br>0.8255<br>0.0579 | 68.7 | 143.4 | 144.3 | 152.2 | 147.1 |
| GoGo | 0.3772<br>0.0174<br>1.1998<br>0.0637 | 69.9 | 148.4 | 149.1 | 156.9 | 151.7 |
| Go | 1.1165<br>0.0286 | 71.6 | 147.2 | 147.4 | 151.5 | 148.9 |

We note the new distribution of TEMO-Go with high potential in Table 2 because it contains the lowest values for standard -LL, AIC, CAIC, BIC and HQIC. Also, through Fig. (2) and Fig. (3), it is evident that the proposed distribution is more identical to this data, and therefore we can choose the distribution this data is best suggested.

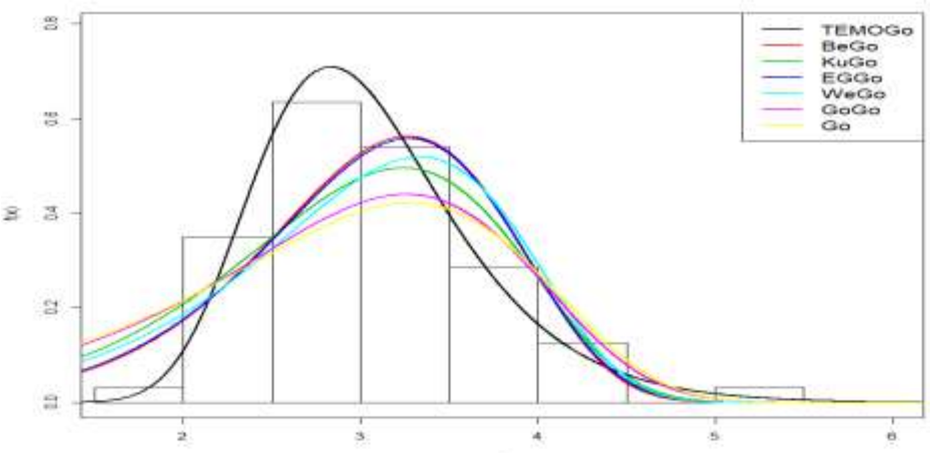


Figure (2): Histogram plot of the dataset1 with the compared distributions

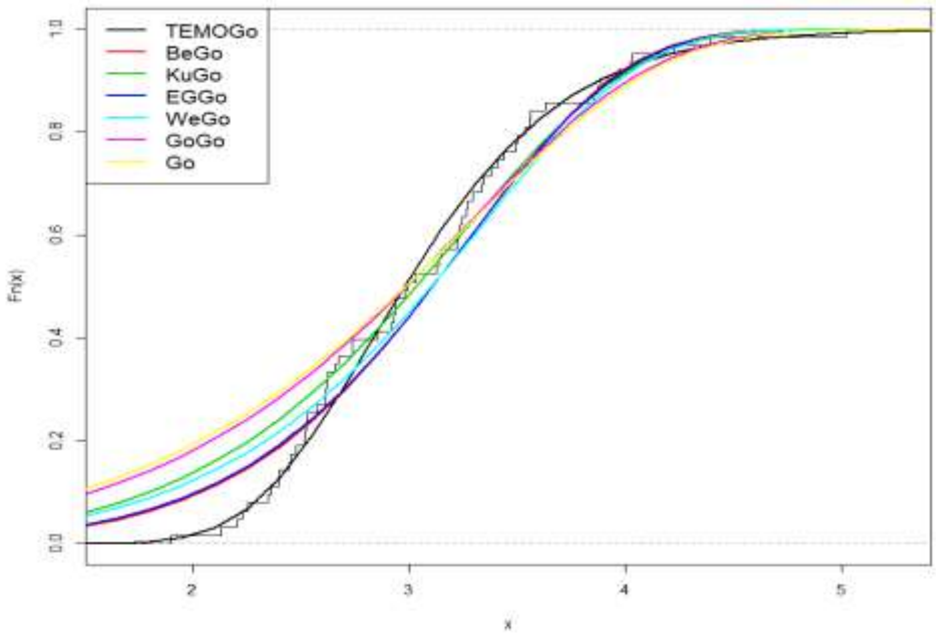


Figure (3):Empirical CDF of the dataset1 with the compared distributions

**6.2. Dataset 2:**

The dataset 2 used reports consists of 63 observations of the strengths of 1.5 cm glass fibers. The dataset 2 has previously been analyzed by (Khaleel et al., 2020), ( Merovci et al., 2016) , (Oguntunde et al., 2017) and (Khaleel et al., 2018) to fit models.

The data are:

| |
|---|
| 0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28,1.29, .48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50,1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63,1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68,1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82,1.84, 1.84, 2.00, 2.01, 2.24 |

Table (3): Summary statistics for dataset 2

| Var | n | mean | sd | median | Min | Max | Skew | Kurtosis |
|---|---|---|---|---|---|---|---|---|
| | 63 | 1.51 | 0.32 | 1.59 | 0.55 | 2.24 | -0.88 | 0.8 |

Note the value of skewness is negative, and this indicates that the data is skewed on the left. As for kurtosis, its value is positive, and this means that the data with thin flatness is close to the normal distribution, meaning that the mean and median of the data are close in value.

Table (4): an estimate of the parameters for the distributions with the statistical criteria
(NLL, AIC, CAIC, BIC, HQIC) for dataset 2

| Model | Est para | -LL | AIC | CAIC | BIC | HQIC |
|---|---|---|---|---|---|---|
| TEMOGo | 1.3915<br>19.217<br>1.4484<br>0.6164 | 12.23 | 32.4 | 33.1 | 40.1 | 35.8 |
| BGo | 2.2388<br>0.7518<br>2.4436<br>0.0975 | 14.5 | 37.1 | 37.7 | 45.6 | 40.4 |
| KuGo | 2.2421<br>0.9024<br>2.3956<br>0.0884 | 14.5 | 37.0 | 37.7 | 45.6 | 40.4 |
| EGGo | 0.8259<br>2.3652<br>2.334737.5<br>0.1106 | 14.5 | 37.1 | 37.7 | 45.7 | 40.5 |
| WGo | 2.4192<br>1.1557<br>1.3250<br>0.1950 | 14.4 | 36.8 | 37.5 | 45.4 | 40.2 |
| GoGo | 0.0742<br>0.9136<br>0.9024<br>0.7291 | 19.5 | 46.9 | 47.6 | 55.5 | 50.3 |
| Go | 2.2685<br>0.0.613 | 23.6 | 51.2 | 51.4 | 55.5 | 52.9 |

We note the new distribution of TEMO-Go with high potential in Table 4 because it contains the lowest values for standard -LL, AIC, CAIC, BIC and HQIC. Also, through Fig. (4) and Fig. (5), it is evident that the proposed distribution is more identical to this data, and therefore we can choose the distribution this data is best suggested.

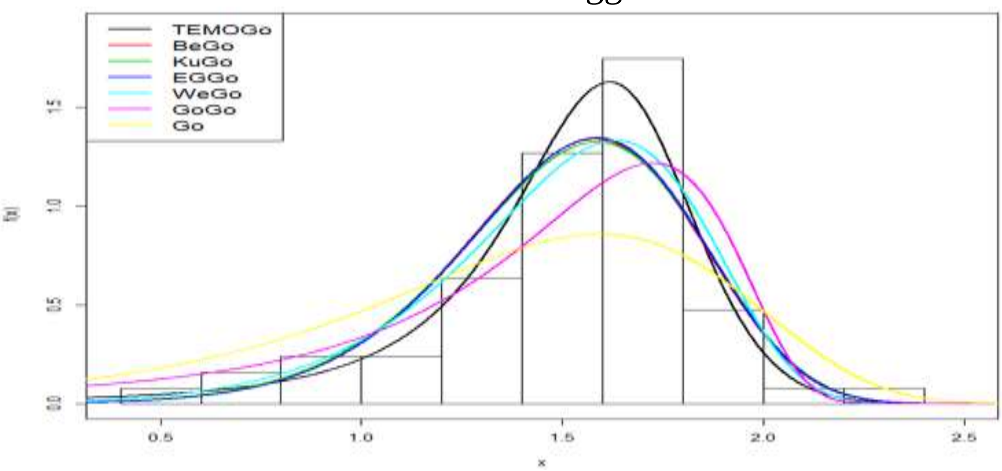


Figure (4): Histogram plot of the dataset 2 with the compared distributions

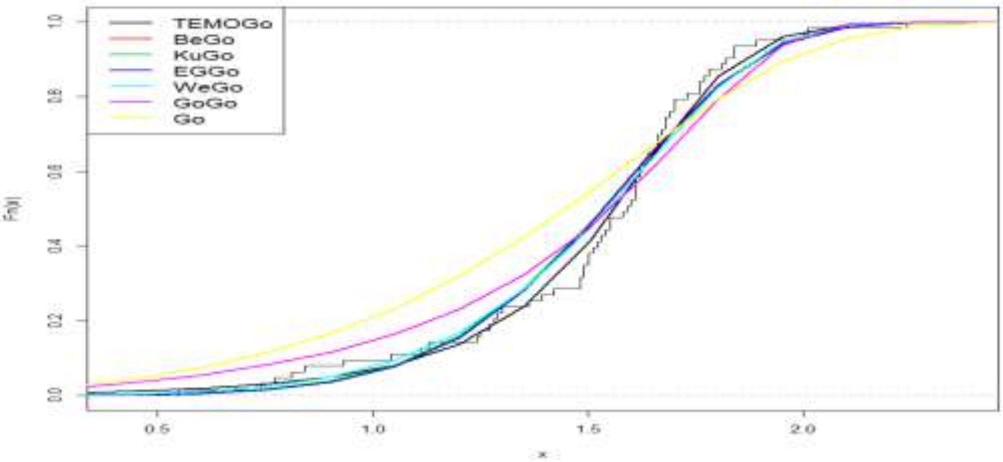


Figure (5): Empirical CDF of the dataset 2 with the compared distributions

## 7. Conclusions:

This paper introduced a new distribution with four-parameter named [0,1]Truncated Exponential Marshall-Olkin-Gompertz (TEMO-Go) .The vital statistical properties of TEMO-Go distribution such that moment, Moment Generating Function , quantile function, order statistics, Entropy. The TEMO-Go parameters are estimated through maximum likelihood estimation method.

Applications with two real data sets (symmetric and right-skewed) are conducted. The newly TEMO-Go distribution has the lowest value of NLL, AIC, CAIC, BIC, and HQIC, so it is shown that it offers better fit and more flexibility than many other comparative models such as Beta Gompertz distribution (BGo), Kumaraswamy Gompertz distribution (KuGo), Exponential Generalized Gompertz distribution (EGGo), Weibull Gompertz distribution (WGo), Gompertz Gompertz distribution (GoGo), Gompertz distribution (Go). This flexibility allows the TEMO-Go distribution to be used in various fields of application.

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