Tikrit Journal of Administrative and Economic Sciences, Vol. 19, No. 61, Part (2): 417-435
Doi: www.doi.org/10.25130/tjaes.19.61.2.22

Tikrit Journal of Administrative and Economics Sciences
مجلة تكريت للعلوم الإدارية والاقتصادية
ISSN: 1813-1719 (Print)

Gompertz Topp–Leone invers Weibull Distributions: Some Properties and Application

Sahib S. Hammed*, Mundher A. Khaleel
Collage of Computer Science and Mathematics, Tikrit University

Keywords:
Gompertz Topp Leone -G family, Moment, Maximum likelihood estimation (MLEs), Invers Weibull distribution.

ARTICLE INFO

Article history:
Received 09 Feb. 2023
Accepted 26 Feb. 2023
Available online 31 Mar. 2023

*Corresponding author:
Sahib S. Hammed
Collage of Computer Science and Mathematics, Tikrit University

Abstract: The Gompertz Topp Leone Inverse Weibull distribution GoTLIW distribution, which has four parameters, is a novel distribution that we introduce in this article. Inverse Weibull (IW) distributions with various parameter values can be combined to form the new distribution. The moments, moment generating function, quantile function, the R'enyi entropy generating function, reliability function, and hazard function are only a few of the crucial structural aspects of the new model that are derived. and apply the method of greatest likelihood to calculate the parameters of the new distribution (MLEs). We used one genuine data point on the total milk output in the first birth of 107 SINDI race cows to demonstrate how adaptable the new distribution is. The Carna 'ba farm, which is owned by Agropecuaria Manoel Dantas Ltd., is the owner of these cows (AMDA), statistics that show how the (GoTLIW) distribution works.

Tikrit Journal of Administrative and Economic Sciences, Vol. 19, No. 61, Part (2): 417-435
Doi: www.doi.org/10.25130/tjaes.19.61.2.22

بعض خصائص توزيع جمبيرتز توب ليون معكوس ويبل مع تطبيق

منذر عبد الله خليل **صاحب صالح حميد**
كلية علوم الحاسوب والرياضيات، جامعة تكريت

المستخلص

في هذا البحث سنقدم توزيعا جديدا يتكون من اربعة معالم ويسمى بتوزيع جمبيرتز توب ليون معكوس ويبل وتمتاز بمرونة عالية وملائمة جيدة لنمذجة البيانات. كما ويمكن دمج معكوس التوزيع ويبل بمعالم أكثر لتوليد العديد من التوزيعات الجديدة. ثم نقوم بتوسيع التوزيع وإعادة كتابة كل من دالة التوزيع التراكمي CDF ودالة الكثافة الاحتمالية PDF لتوزيع جديد وايضا يعطي التوزيع الجديد اشكال بيانية لدالة الكثافة الاحتمالية وتكون ممتدة أكثر او الملتوية لليسار. كما نقدم العديد من الخصائص الاحصائية للتوزيع الجديد مثل العزوم والدالة المولدة للعزوم والدالة التجزيئية والدالة البقاء ودالة المخاطرة وبعض الدوال الاخرى. وكذلك تم تطبيق طريقة الامكان الاعظم لتقدير معالم التوزيع الجديد. ولبيان مدى ملائمة ومرونة التوزيع الجديد تم استخدام مجموعة من البيانات الحقيقية التي تتعلق بأجمالي انتاج حليب في اول ولادة (107) بقرة من سلالة SINDI وحيث اثبت التوزيع الجديد ملائمة عالية ومرونة جيدة في نمذجة للبيانات الحقيقية بالمقارنة مع بعض التوزيعات المعروفة باستخدام بعض المعايير الاحصائية.

الكلمات المفتاحية: عائلة جمبيرتز توب ليون، العزوم، دوال مولدة للعزوم، تقدير المعالم باستخدام دالة الإمكان الأعظم، توزيع معكوس ويبل.

1. Introduction

Nowadays, in probability theory, both the reliability function and the survival function have the same property, which is the measurement of the life span of a particular system or living organism. The flexible distribution is considered to fit different lifetime data. These distributions have more flexibility than the base line distribution because they add one or more shape parameters. Furthermore, new distribution can take many different shapes. Many families and distributions have been established and studied by researchers. In (1993) Mudholkar and Srivastava proposed a modification of the Weibull family named Inverse Weibull distribution (IW) by adding one shape parameter. In (1994) Mudholkar et al., Generalized Weibull family. In (2021, March) Chipepa et al., Marshal Olkin Gompertz distribution. (1997) developed a technique for adding one parameter to the baseline distribution, and the new distribution is very flexible for modeling real time data. in (2019). The Marshall-Olkin inverse Lomax distribution (MO-ILD) with application .in (2021) A new inverse Weibull distribution: properties, classical and Bayesian estimation with applications. In (2002) Eugene et al.

Doi: www.doi.org/10.25130/tjaes.19.61.2.22

introduced a new technique by adding two shape parameters named Beta-G. (2017). Beta burr type x with application to rainfall data. Malaysian Journal of Mathematical Sciences. Bera et al., (2015) The Kumaraswamy inverse Weibull Poisson distribution, (2017). WEIBULL BURR X DISTRIBUTION. In (2013) Cordeiro et al. introduced a new family using an odd formula to define Weibull-G. In (2017) Weibull Burr X distribution. (2012). Modified inverse Weibull distribution. Abid et al. proposed a new method in (2017) to discover a new family called the [0,1] truncated-G family. In (2022). A new [0, 1] truncated inverse Weibull Rayleigh distribution. In (2017), Alizadeh et al. defined a new family named Gompertz-G by adding two shape parameters to any base line distribution. Many researchers follow the work by Alizadeh et al. to define new distributions like. in (2020) The Gompertz flexible Weibull distribution. in (2022) the Rayleigh Gompertz distribution. Muhammed, H. Z., & Almetwally, E. M. (2020). Bayesian and non-Bayesian estimation for the bivariate inverse Weibull distribution. Pakungwati al., Marshall-Olkin extended inverse Weibull distribution and its application (2018, November). In 2021, Khaleel et al. defined and studied another new family called Marshall Olkin Topp-Leone-G a day in probability theory, the flexible distribution considers to fit different the lifetime data. This distribution will be more flexibility than the base line distribution because we add one or more shape parameters as we can proof that in experimental application. Furthermore, new distribution can give many different shapes. In (2020) the Marshall-Olkin Topp Leone-G family of distributions. Finally, The Gompertz Tope Leone-G (GoTL-G) family of distributions was presented by Rannona in (2022). Therefore, the cumulative density function (cdf) and the probability density function (pdf) of the GoTL-G family of distributions is given by.

$$F_{GOTL-G}(x;\gamma,b,\zeta) = \left(1 - e^{\left\{\frac{1}{\gamma}(1-(1-[1-\bar{G}^2(x,\zeta)]^b)^{-\gamma}\right\}}\right) \quad (1)$$

Where $\bar{G}(x;\zeta) = 1 - G(x;\zeta) \Rightarrow \bar{G}^2(x;\zeta) = (1 - G(x;\zeta))^2$

$$f_{GOTL-G}(x;\gamma,b,\zeta) = 2\,b\,\mathrm{g}(\mathrm{x};\zeta)\overline{\mathrm{G}}(\mathrm{x};\zeta)\,(1 - \overline{\mathrm{G}}^2(x;\zeta))^{b-1} * [1 - (1 - \overline{\mathrm{G}}^2(\mathrm{x};\zeta)^b]^{-\gamma-1} \quad (2)$$

$$* \left(e^{\left\{\frac{1}{\gamma}(1-(1-[1-\bar{G}^2(x,\zeta)]^b)^{-\gamma}\right\}} \right)$$

therefore $x > 0$, $\gamma, \text{b} > 0$ and ζ is the parameter vector.

We are motivated to propose a distribution because:

- It can be considered as an appropriate distribution to model the skewed data available in the literature may not be adequately fitted.
- It may also be used to represent a wide range of real-world data sets in the disciplines of survival and industrial reliability.
- It can be relied upon to build other unstudied distributions with better results and with the same or other data.

2. **Inverse Weibull Distribution**: The inverse Weibull distribution has a novel addition known as the extension, which was developed by the researchers Afify et al. in (2021) Khan et al., in (2008).

A novel inverse Weibull distribution was investigated, and some of its mathematical characteristics were deduced. A variety of techniques, including the method of greatest possibility and the method of least squares, were used to estimate the parameters of the new distribution. The inverse Weibull functions are both writable. If x is a random variable, then $G(\text{x};\zeta)$is the probability distribution function (pdf), and $\text{g}(\text{x};\zeta)$is the probability density function (pdf) corresponding to the inverse Weibull distribution (cdf):

$$\text{G}(\text{x};\alpha,\beta) = \text{e}^{-\alpha(\text{x})^{-\beta}} \qquad \text{x} > 0 \quad ,\alpha,\beta > 0 \qquad (3)$$

$$\text{g}(\text{x};\alpha,\beta) = \text{a}\beta\text{e}^{-\text{a}(\text{x})^{-\beta}}(\text{x})^{-\beta-1} \quad \text{x} > 0 \quad ,\alpha,\beta > 0 \qquad (4)$$

3. **Gompertz Toppe Leone invers Weibull distribution:** In this paper, we develop a novel distribution called the Gompertz Toppe Leone invers Weibull distribution that is the outcome of combining the Gompertz Toppe Leone family with the Invers Weibull distribution. By contrasting it with certain different distributions, it is hoped that we may find some useful conclusions. Further, we do. Consequently, we change equation one into equation three to obtain the cumulative distribution function. By inserting Eq. (3) in Eq. (1) we have a cdf of new distribution as follows:

$$F_{GOTLIW}(x;\gamma,b,\beta,a) = \left[1 - e^{\left\{\frac{1}{\gamma}\left[(1-(1-\left[1-\left(1-e^{-a(x)^{-\beta}}\right)^2\right]^b)^{-\gamma})\right]\right\}}\right] \quad (5)$$

Therefore, we substitute equation number(4) into equation number (2) to obtain the probability density function:

$$f_{GoTLIW}(x;\gamma,b,\beta,a) = 2ab\beta e^{-a(x)^{-\beta}}(x)^{-\beta-1}\left[1-e^{-a(x)^{-\beta}}\right] * [1 - \left(1-e^{-a(x)^{-\beta}}\right)^2]^{b-1} * \left[1-(1-\left(1-e^{-a(x)^{-\beta}}\right)^2)^b\right]^{-\gamma-1} * \left(e^{\left\{\frac{1}{\gamma}(1-(1-\left[1-\left(1-e^{-a(x)^{-\beta}}\right)^2\right]^b)^{-\gamma})\right\}}\right) \quad (6)$$

Where $x > 0,\ \gamma, b, \beta, a > 0$

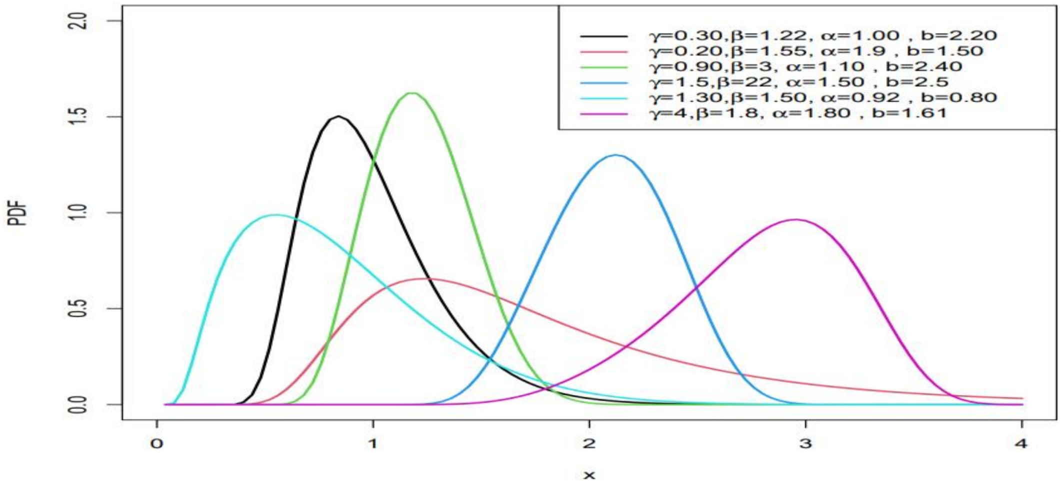


Figure 1: Different shapes of the pdf, with different value of parameters.

3-1. The survival function S(x) given by:

$$S_{GOTLIW}(x;\gamma,b,\beta,a) = e^{\left\{\frac{1}{\gamma}\left[(1-(1-\left[1-\left(1-e^{-a(x)^{-\beta}}\right)^2\right]^b)^{-\gamma})\right]\right\}} \quad (7)$$

3-2. Hazard rate function (HRF) by substituting:

Tikrit Journal of Administrative and Economic Sciences, Vol. 19, No. 61, Part (2): 417-435
Doi: www.doi.org/10.25130/tjaes.19.61.2.22

$$h_{GOTLIW}(x;\gamma,b,\beta,a)$$
$$=\left[\frac{2ab\beta e^{-a(x)^{-\beta}}(x)^{-\beta-1}\left[1-e^{-a(x)^{-\beta}}\right]\left[1-\left(1-e^{-a(x)^{-\beta}}\right)^{2}\right]^{b-1}\left[1-\left(1-\left(1-e^{-a(x)^{-\beta}}\right)^{2}\right)^{b}\right]^{-\gamma-1}}{e^{\left\{\frac{1}{\gamma}\left[(1-(1-\left[1-\left(1-e^{-a(x)^{-\beta}}\right)^{2}\right]^{b})^{-\gamma})\right]\right\}}}\right]$$
$$*$$
$$\left[e^{\left\{\frac{1}{\gamma}\left[(1-(1-\left[1-\left(1-e^{-a(x)^{-\beta}}\right)^{2}\right]^{b})^{-\gamma})\right]\right\}}\right]$$

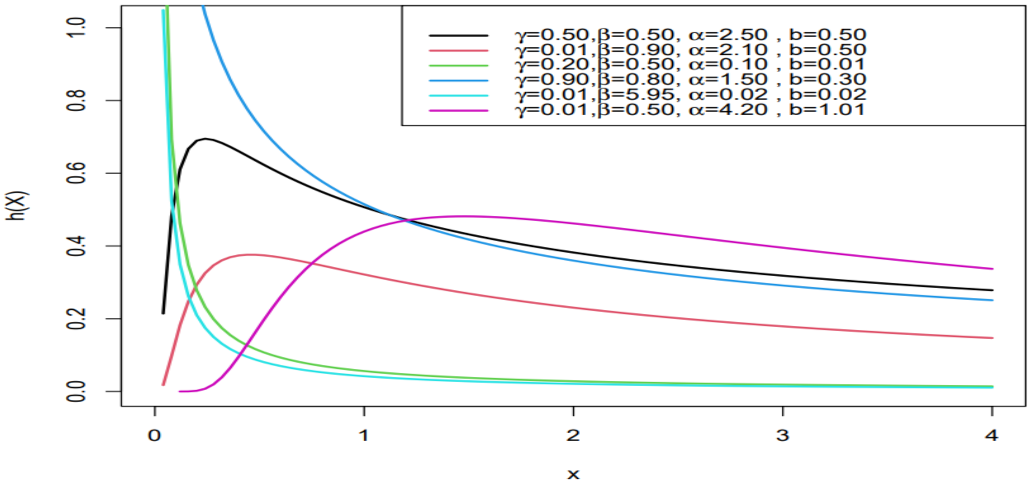


Figure (2): Different shapes of the h(x), function with different value of parameters

4. Expansion the density function: The expansion for the pdf of the GoTLIW distribution are provided as:

Let us use the exponential Taylor series $e^{-x}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\,x^{i}}{i!}$ for expansion for the pdf of new distribution on Eq. (6).

$$M_x(t)=E(e^{tx})=\int_{-\infty}^{\infty}e^{tx}f(x)dx$$

$$f_{GoTLIW}(x;\gamma,b,\beta,a)=2ab\beta(q+1)\ (x)^{-\beta-1}.e^{-a(q+1)(x)^{-\beta}}*$$
$$\sum_{k,L,M,p,q=0}^{\infty}\frac{(-1)^{L+M+p+q}}{k!\,\gamma^{k}(q+1)}(k\ L\ )\ (-\gamma(L+1)-1\ M\ )(b(M+1)-1\ p\ )(1$$
$$+2p\ q\ )\qquad\qquad(9)$$

Equation (9) can be written as a short form and it help us to find many properties than need many simplified before find the integration.

Doi: www.doi.org/10.25130/tjaes.19.61.2.22

$$f_{GoTLIW}\,(x;\gamma,b,\beta,a) = \mho a\beta(q+1)\,(x)^{-\beta-1} * e^{-a(q+1)(x)^{-\beta}}$$

$$\sum_{k,L,M,p,q=0}^{\infty} \frac{(-1)^{L+M+p+q}}{k!\,\gamma^{k}(q+1)} \binom{k}{L} \binom{-\gamma(L+1)-1}{M} \binom{b(M+1)-1}{p} \binom{1+2p}{q} \qquad (10)$$

Where $\mho$

$$= \sum_{k,L,M,p,q=0}^{\infty} \frac{(-1)^{L+M+p+q}}{k!\,\gamma^{k}(q+1)} \binom{k}{L} \binom{-\gamma(L+1)-1}{M} \binom{b(M+1)-1}{p} \binom{1+2p}{q}$$

Equation (10) as we shall demonstrate, it may be utilized to discover a variety of mathematical features.

5. Mathematical Properties:

5-1. Moments: One of the statistical techniques utilized in these situations is mathematical expectation. Moments are crucial in identifying and quantifying certain statistical features. Any attribute, including the coefficient of variation, torsion and flattening, standard deviation, as well as the potential of determining the mean, variance, and other metrics, may be investigated using moments. Depending on the probability density function after its expansion, we may get the moment of degree r for the Gompertz Toppe Leone inverses Weibull distribution from the following relationship; [Khalaf and Khaleel (2022)]

$$\mu_r = E(x^r) = \int_0^{\infty} x^r f(x)dx$$

$$\mu_r = E(x^r)_{GoTLIW} = \mho \int_0^{\infty} x^r (\mathrm{x})^{-\beta-1}\, a\beta(q+1)e^{-a(q+1)(\mathrm{x})^{-\beta}} dx \qquad (11)$$

Making substitution $z = a(q+1)(x)^{-\beta} \Longrightarrow dz = -a\beta(q+1)(x)^{-\beta-1}dx$

$$zx^{\beta} = a(q+1) \quad \Longrightarrow x^{\beta} = a(q+1)z^{-1} \Longrightarrow x = \big(a(q+1)\big)^{\frac{1}{\beta}} z^{\frac{-1}{\beta}}$$

Putting these values, the expression will be

$$\mu_r = E(X^r)_{GoTLIW} = \mho \int_0^{\infty} \left(\big(a(q+1)\big)^{\frac{1}{\beta}} z^{\frac{-1}{\beta}}\right)^{r} e^{-z}dz$$

Doi: www.doi.org/10.25130/tjaes.19.61.2.22

$$\mu_r = E(X^r)_{GoTLIW} = \mho\left(a(q+1)\right)^{\frac{r}{\beta}} \int_0^\infty z^{\frac{-r}{\beta}} e^{-z} dz$$

$$\mu_r = E(X^r)_{GoTLIW} = \mho\left(a(q+1)\right)^{\frac{r}{\beta}} \int_0^\infty z^{\left(1-\frac{r}{\beta}\right)-1} e^{-z} dz \quad \text{the integral of the expression is}$$

$$= \mho[a(q+1)]^{\frac{r}{\beta}} \Gamma\left(1 - \frac{r}{\beta}\right) \tag{12}$$

Equation (12) very important to find many statistical concepts like mean, the 4th moments, variance, CV, Moment generating function (MGF) and so on.

5-2. Moments Generating Function:

By using Eq. (12) and Taylor expansion $e^{tx} = \sum_{\rho=0}^{\infty} \frac{(t)^\rho}{\rho!} x^\rho$ we can get the MGF of new distribution as follows: [Khalaf and Khaleel. (2020)]

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$M_x(t) = \sum_{\rho=0}^{\infty} \frac{t^\rho}{\rho!} \int_0^\infty x^\rho f(x) dx = \sum_{\rho=0}^{\infty} \frac{t^\rho}{\rho!} E(X^\rho) = \sum_{\rho=0}^{\infty} \frac{t^\rho}{\rho!} [E(X^\rho)]$$

By using Eq. (12), it is same moment's function we have

$$M_x(t) = \sum_{\rho=0}^{\infty} \frac{t^\rho}{\rho!} \left[\mho[a(q+1)]^{\frac{r}{\beta}} \Gamma\left(1 - \frac{r}{\beta}\right)\right] \tag{13}$$

5-3. The Quantile function and median: We get the quantity function using equation no. (6), say, x = Q(u), [Khaleel et al., (2017)] $u = F^{-1}(x)$

$$1 - u = e^{\left\{\frac{1}{\gamma}\left[(1-(1-\left[1-\left(1-e^{-a(x)^{-\beta}}\right)^2\right]^b)^{-\gamma})\right]\right\}}$$

$$Q(x)_{GoTLIW} = \left(a^{-1}\log 1 - \left[1 - \left(1 - \left((1-\gamma\log(1-u))\right)^{\frac{1}{-\gamma}}\right)^{\frac{1}{b}}\right]^{\frac{1}{2}}\right)^{\frac{1}{\beta}} ; 0 < u < 1 \tag{14}$$

Equation (14) very important to study simulation of the GoTLIW distribution where U is $Uniform$ (0,1).

Tikrit Journal of Administrative and Economic Sciences, Vol. 19, No. 61, Part (2): 417-435
Doi: www.doi.org/10.25130/tjaes.19.61.2.22

Hence, the median of the distribution is derived by substituting $u = 0.5$ in Equation (14)

$$\text{Median}(x)_{\text{GoTLIW}} = \left(a^{-1}\log 1 - \left[1-\left(1-\left((1-\gamma\log(1-0.5))\right)^{\frac{1}{-\gamma}}\right)^{\frac{1}{b}}\right]^{\frac{1}{2}} \right)^{\frac{1}{\beta}} \qquad (15)$$

5-4. Order Statistics: Let $X_1, X_2, X_3, \ldots, X_n$ denote a random sample of size n drawn from the GoTLIW distribution, and $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ denote the order statistics. If $X_{i:n}$ denotes statistics of the ith order, then the probability density function of $X_{i:n}$ is given by. [Munef and Khaleel (2021)].

$$g_{i:n}(x) = \sum_{j=0}^{n-i} (-1)^j \frac{n!}{(i-1)!\,(n-i)!} \binom{n-i}{j} [F(x)]^{j+i-1} f(x) \qquad (16)$$

$$g_{i:n_{GoTLIW}}(x) = \sum_{j=0}^{n-i} \frac{n!}{(i-1)!\,(n-i)!} (-1)^j \binom{n-i}{j} * \left[\left(1 - e^{\left\{\frac{1}{\gamma}(1-(1-\left[1-\left(\left(1-\mathrm{e}^{-a(\mathrm{x})^{-\beta}}\right)^2\right)\right]^b)^{-\gamma})\right\}}\right)\right]^{j+i-1} *$$
$$2ab\beta \mathrm{e}^{-\mathrm{a}(\mathrm{x})^{-\beta}} (\mathrm{x})^{-\beta-1} \left[1-\mathrm{e}^{-a(\mathrm{x})^{-\beta}}\right] ** [1-\left(1-\mathrm{e}^{-a(\mathrm{x})^{-\beta}}\right)^2]^{b-1}$$
$$* \left[1-\left(1-\left(1-\mathrm{e}^{-a(\mathrm{x})^{-\beta}}\right)^2\right)^b\right]^{-\gamma-1} * \left(e^{\left\{\frac{1}{\gamma}(1-(1-\left[1-\left(1-\mathrm{e}^{-a(\mathrm{x})^{-\beta}}\right)^2\right]^b)^{-\gamma})\right\}}\right) \qquad (17)$$

$$g_{i:n}(x) = \sum_{j=0}^{n-i} \frac{n!}{(i-1)!\,(n-i)!} \binom{n-i}{j} \sum_{k}^{\infty}\sum_{p}^{\infty}\sum_{q}^{\infty}\sum_{r}^{\infty}\sum_{s}^{\infty} \frac{(-1)^{k+p+q+r+s}}{p!\,\gamma^p} \binom{j+i-1}{k} \binom{k+1}{p} \binom{p}{q} \binom{-\gamma(q+1)-1}{r} *$$
$$\binom{b(r+1)-1}{s} *. \binom{2s+1}{z} 2ab\beta \mathrm{e}^{-\mathrm{a}(\mathrm{z}+1)(\mathrm{x})^{-\beta}} (\mathrm{x})^{-\beta-1} \qquad (18)$$

The density function of the smallest order statistics is obtained by Putting r=1 and, Similarly, the density function of the greatest order statistic can be obtained by substituting r = n.

5-5. R'enyi Entropy: The study of life and water resources makes extensive use of the R'enyi Entropy. It is a fundamental number in information theory related to any random variable and is significant in spectrum analysis and model performance assessment. It may also be thought of as an average level. The extended frequency analysis is a metric for the level of uncertainty that is sometimes referred to in randomized trials as "unproven." It is also applied in the areas of information theory and statistical physics. The probability density function for a novel Gompertz-Topp-Leone invers Weibull distribution, which was examined in this thesis, defines R'enyi Entropy for the random variable x. Consequently, this is the conventional formula: [Munef et al., (2021)].

$$I_R(S) = \frac{1}{1-s}\log\int_0^\infty ([f(x)]^s dx) \qquad s \neq 1 \;, s > 0.$$

$$I_R(S) = \frac{1}{1-s}\log\int_0^\infty [f_{GOTLIW}(x;\gamma,b,\beta,a)]^s dx \qquad (19)$$

$a, b, \gamma, \beta > 0$ and $x > 0$, $s > 0$Where

And the probability density function for a new GoTLIW distribution. is as follows;

$$f_{GOTLIW}(x;\gamma,b,\beta,a) = 2ab\beta e^{-a(x)^{-\beta}}(x)^{-\beta-1}\left[1-e^{-a(x)^{-\beta}}\right] * \left[1-\left(1-e^{-a(x)^{-\beta}}\right)^2\right]^{b-1} * \left[1-\left(1-\left(1-e^{-a(x)^{-\beta}}\right)^2\right)^b\right]^{-\gamma-1} * \left(e^{\left\{\frac{1}{\gamma}(1-(1-\left[1-\left(1-e^{-a(x)^{-\beta}}\right)^2\right]^b)^{-\gamma})\right\}}\right)$$

Thus, the formula is the final R'enyi Entropy for a new distribution, and the number of the equation (19) is as follows;

$$I_R(S) = \frac{1}{1-s}\log\int_0^\infty \left[2ab\beta e^{-a(x)^{-\beta}}(x)^{-\beta-1}\left[1-e^{-a(x)^{-\beta}}\right] * [1 - \left(1-e^{-a(x)^{-\beta}}\right)^2]^{b-1} * \left[1 - (1 - \left(1-e^{-a(x)^{-\beta}}\right)^2)^b\right]^{-\gamma-1} \left(e^{\left\{\frac{1}{\gamma}(1-(1-\left[1-\left(1-e^{-a(x)^{-\beta}}\right)^2\right]^b)^{-\gamma})\right\}}\right)\right]^s dx \quad (20)$$

$$I_R(S) = \left[\mathcal{M}\ (a^s\beta^s\frac{-1}{\beta}(a(s+p))^{\frac{-s\beta-s}{\beta}+\frac{1}{\beta}}\right]\Gamma\left(\frac{s\beta+s-1}{\beta}\right) \qquad s \neq 1, \quad s > 0. \qquad (21)$$

Equation (21) very important in this section when $S \Longrightarrow 0$ we have Shannon Entropy.

5-6. Parameter Estimation of GoTLIW distribution: To determine the unknown parameters of the GoTLIW distribution, we take into account the maximum likelihood estimation approach. The sample values, which are composed of n observations, are x_1, x_2, x_3,.., x_n. The probability density function's log-likelihood function is provided by.

$$EP = \prod_{i=1}^{n} f_{GOTLIW}\,(x;\gamma,b,\beta,a) \qquad (22)$$

where E is Estimation and P is Parameter

$$EP = \prod_{i=0}^{n}\left[\left(2ab\beta e^{-a(x)^{-\beta}}(x)^{-\beta-1}\left[1-e^{-a(x)^{-\beta}}\right] * [1 - \left(1-e^{-a(x)^{-\beta}}\right)^2]^{b-1}\right) * \left[1-(1-\left(1-e^{-a(x)^{-\beta}}\right)^2)^b\right]^{-\gamma-1} * \left(e^{\left\{\frac{1}{\gamma}(1-(1-\left[1-\left(1-e^{-a(x)^{-\beta}}\right)^2\right]^b)^{-\gamma})\right\}}\right)\right]$$

$$\ln(EP) = n\log(2ab\beta) - (\beta+1)\sum_{i10}^{n}\ln(x) - ax_i^{-\beta} + \sum_{i=1}^{n}\ln\left(1-e^{-ax_i^{-\beta}}\right) + (b-1)\sum_{i=1}^{n}\ln\left(1-\left(1-e^{-ax_i^{-\beta}}\right)^2\right) - (\gamma+1)\sum_{i=1}^{n}\ln\left(1-\left[(1-\left(1-e^{-ax_i^{-\beta}}\right)^2\right]^b\right) + \sum_{i=1}^{n}\frac{1}{\gamma}\left(1-(1-(1-\left(1-e^{-ax_i^{-\beta}}\right)^2)^b)^{-\gamma})\right) \quad (23)$$

Now, we extract the partial derivatives for each parameter, thus:

$$\frac{\partial(EP)}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\frac{a\ln(x_i)}{x_i^{\beta}} - \sum_{i=1}^{n}\frac{a\ln(x_i)e^{-a(x_i)^{-\beta}}}{x_i^{\beta}\left(1-e^{-a(x_i)^{-\beta}}\right)} + \sum_{i=1}^{n}\frac{2a(b-1)\ln(x)e^{-a(x)^{-\beta}}\left(1-e^{-ax_i^{-\beta}}\right)}{x^{\beta}\left(1-(1-e^{-a(x_i)^{-\beta}})^2\right)}$$
$$-\sum_{i=1}^{n}\frac{2ab(\lambda+1)\ln(x_i)e^{-a(x_i)^{-\beta}}\left(1-e^{-ax_i^{-\beta}}\right)\left(1-\left(1-e^{-a(x_i)^{-\beta}}\right)^2\right)^{b-1}}{x_i^{\beta}\left(1-\left(1-(1-e^{-a(x_i)^{-\beta}})^2\right)^b\right)}$$
$$+\sum_{i=1}^{n}\frac{2ab\ln(x_i)e^{-a(x_i)^{-\beta}}\left(1-e^{-ax_i^{-\beta}}\right)\left(1-\left(1-e^{-a(x_i)^{-\beta}}\right)^2\right)^{b-1}\left[1-\left(1-\left(1-e^{-a(x_i)^{-\beta}}\right)^2\right)^{b-1}\right]^{-\gamma-1}}{x_i^{\beta}} \quad (24)$$

$$\frac{\partial(EP)}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n}\frac{1}{x_i^{\beta}} - \frac{\sum_{i=1}^{n}\left(x_i^{-\beta}e^{-a(x_i)^{-\beta}}\right)}{n+\sum_{i=1}^{n}\left(-e^{-a(x_i)^{-\beta}}\right)} + \sum_{i=1}^{n}\frac{2(b-1)\left(e^{-a(x_i)^{-\beta}}\right)\left(1-e^{-a(x_i)^{-\beta}}\right)}{x_i^{\beta}\left(1-\left(1-e^{-a(x_i)^{-\beta}}\right)^2\right)}$$
$$-(\lambda+1)\sum_{i=1}^{n}\frac{2e^{-a(x_i)^{-\beta}}b\left(1-e^{-a(x_i)^{-\beta}}\right)x_i^{-\beta}\left(1-\left(1-e^{-a(x_i)^{-\beta}}\right)^2\right)^{b-1}}{\left(n-\sum_{i=1}^{n}\left(1-\left(1-e^{-a(x_i)^{-\beta}}\right)^2\right)^b\right)}$$
$$-\sum_{i=1}^{n}\frac{2b\,e^{-a(x_i)^{-\beta}}\left(1-e^{-a(x_i)^{-\beta}}\right)\left(1-\left(1-e^{-a(x_i)^{-\beta}}\right)^2\right)^{b-1}\left[1-\left(1-(1-e^{-a(x_i)^{-\beta}})^2\right)^{b-1}\right]^{-\gamma-1}}{x_i^{\beta}} \quad (25)$$

$$\frac{\partial(EP)}{\partial b} = \frac{n}{b} - \sum_{i=1}^{n}\ln\left(1-\left(1-e^{-a(x_i)^{-\beta}}\right)^2\right) - \sum_{i=1}^{n}\frac{(\gamma+1)\ln\left[1-\left(1-e^{-a(x_i)^{-\beta}}\right)^2\right]\left[1-\left(1-e^{-a(x_i)^{-\beta}}\right)^2\right]^b}{\left(1-\left[1-(1-e^{-a(x_i)^{-\beta}})^2\right]^b\right)}$$

Doi: www.doi.org/10.25130/tjaes.19.61.2.22

$$+\sum_{i=1}^{n}\left(1-(1-\left[(1-(1-e^{-a(x_i)^{-\beta}})^2\right]^b)^{-\gamma-1}\right)\ln\left(\left(1-e^{-a(x_i)^{-\beta}}\right)^2\right)\left[(1-\left(1-e^{-a(x_i)^{-\beta}}\right)^2\right]^b \quad (26)$$

$$\frac{\ln(EP)}{\partial\gamma}=-\sum_{i=1}^{n}\ln\left(1-\left(1-\left(1-e^{-ax_i^{-\beta}}\right)^2\right)^b\right)+$$

$$\sum_{i=1}^{n}\frac{\ln\left(1-(1-\left(1-e^{-ax_i^{-\beta}}\right)^2)^b\right)\gamma-\left(1-(1-\left(1-e^{-ax_i^{-\beta}}\right)^2)^b\right)^\gamma+1}{\left(1-(1-\left(1-e^{-ax_i^{-\beta}}\right)^2)^b\right)^\gamma\gamma^2} \quad (27)$$

Following a series of steps, we observe that the equations are zeroed to get the largest estimate possible: $\frac{\partial(EP)}{\partial\beta}=\frac{\partial(EP)}{\partial\gamma}=\frac{\partial(EP)}{\partial b}=\frac{\partial(EP)}{\partial a}=0$

The equations have the integers (24), (25), (26), and (27) and are equivalent to zero after that. It displays the equations, but it is not able to access the simplified formula to estimate the parameters in it, making it challenging to manually solve. Therefore, it is important to get an approximation of these parameters for the distribution of the new GoTLIW using computer programs, numerical methods, or by using matrices. In order to get the estimate of these parameters, the R software was utilized, which is one of the crucial and simple statistical programs. to support the new Gompertz-Topp-Leone Invers Weibull distribution**.**

6. Applications: In this part, we provide a real-world phenomenon for the GOTLIW distribution, which also fits better than other distributions. Their (NLL) negative log-likelihood is included in the comparison, (HQIC) Hanan and Quinn Information Criteria, (BIC) Bayesian Information Criteria, (CAIC) Consistent Akaike Information Criteria, (AIC) Akaike Information Criteria values, Kolmogorov-Smirnov (KS), P-Value and. The data fitting comparison between the GoTLIW distribution and other distributions such as, Transmuted Exponentiated Exponential inverse Weibull distribution, The Beta Inverse Weibull distribution, The Kumaraswamy–Inverse Weibull distribution, Exponentiated Generalized inverse Weibull distribution, Weighted Inverse Weibull Distribution, Gompertz inverse Weibull distribution, Marshall-Olkin Extended Inverse Weibull Distribution, Invers Weibull Distribution,

The data: Is about the total milk production in the Örst birth of (107) cows from SINDI race. These

Doi: www.doi.org/10.25130/tjaes.19.61.2.22

cows are property of the Carna˙ba farm which belongs to the Agropecu·ria Manoel Dantas Ltda (AMDA), located in Tapero· City, Paraiba (Brazil). Hamedani et al., (2017). The original data is not in the interval (0,1), and it was necessary to make a transformation given by:

xi = [yi-min(yi)]/[max(yi)-min(yi)], for i=1,…,107.

The values of yi are given in Table of Brito (2009, p. 46) and xi values are 0.4365, 0.4260, 0.5140, 0.6907, 0.7471, 0.2605, 0.6196, 0.8781, 0.4990, 0.6058, 0.6891, 0.5770, 0.5394, 0.1479, 0.2356, 0.6012, 0.1525, 0.5483, 0.3323, 0.0671, 0.2361, 0.4800, 0.5707, 0.7131, 0.5853, 0.6768, 0.5350, 0.4151, 0.6789, 0.4576, 0.3259, 0.2303, 0.7687, 0.4371, 0.3383, 0.6114, 0.3480, 0.4564, 0.7804, 0.3406, 0.4823, 0.5912, 0.5744, 0.5481, 0.1131, 0.7290, 0.0168, 0.5529, 0.4530, 0.3891, 0.4752, 0.3134, 0.3175, 0.1167, 0.6750, 0.5113, 0.5447, 0.4143, 0.5627, 0.5150, 0.0776, 0.3945, 0.4553, 0.4470, 0.5285, 0.5232, 0.6465, 0.0650, 0.8492, 0.8147, 0.3627, 0.3906, 0.4438, 0.4612, 0.3188, 0.2160, 0.6707, 0.6220, 0.5629, 0.4675, 0.6844, 0.3413, 0.4332, 0.0854, 0.3821, 0.4694, 0.3635, 0.4111, 0.5349, 0.3751, 0.1546, 0.4517, 0.2681, 0.4049, 0.5553, 0.5878, 0.4741, 0.3598, 0.7629, 0.5941, 0.6174, 0.6860, 0.0609, 0.6488, 0.2747).

The new distribution: Gompertz Toppe Leone invers Weibull distribution was compared with some sub-models:

And The R program was used to calculate the estimation values of the new distribution parameters using the MLE method. also, the most important statistical measures that are used in the comparison between the distributions, which were mentioned in the first chapter in detail values AIC, CAIC, BIC, HQBIC, Kolmogorov-Smimov (KS) and p-value test statistic.

Table (1): The measures AIC, CAIC, BIC, HQIC and, Criteria Values

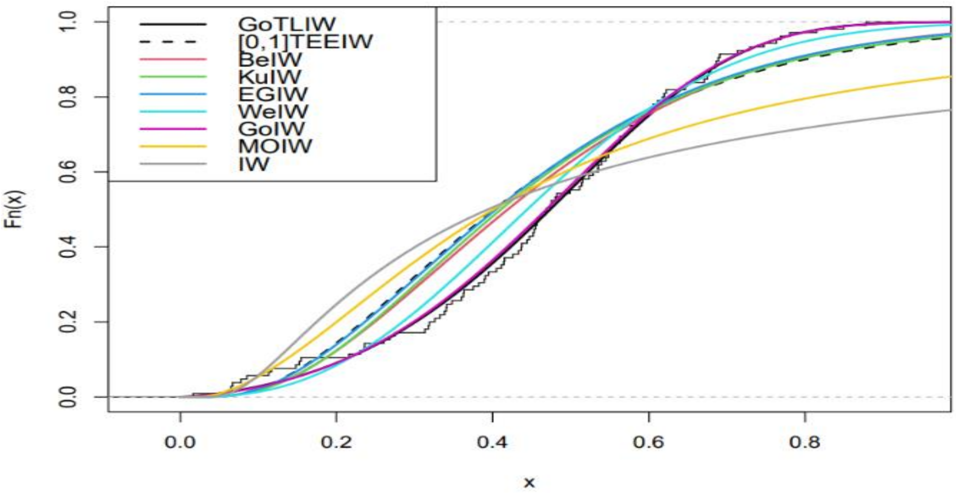
Quality statistical standards with a new distribution. GoTLIW						
Distributions	Criteria Values		AIC	CAIC	BIC	HQIC
	KS Statistic	KS p-value				
GoTLIW	0.0473	0.9728	-48.8643	-48.464	-38.248	-44.562
TEEIW	0.1730	0.0037	1.31893	1.718	11.934	5.620
BIW	0.1433	0.0267	-7.5425	-7.142	3.0732	-3.240
KUIW	0.1580	0.0105	-5.2255	-4.825	5.3903	-0.923
EGIW	0.1696	0.0047	-3.4558	-3.055	7.1599	0.845
WeIW	0.0972	0.2734	-31.253	-30.853	-20.637	-26.952
GoIW	0.0494	0.9598	-48.125	-47.725	-37.509	-43.823
MOIW	0.2091	0.0002	37.736	37.973	45.697	40.962
IW	0.2609	1.2311e-06	97.5650	97.682	102.872	99.715

Source: Prepared by the researcher based on statistical software

Table (2): The estimated feature value of the distributions for the real data set.

Distributions	Estimation parameters				
	-LL	$\hat{\gamma}$	$\hat{b}$	$\hat{a}$	$\hat{\beta}$
GoTLIW	-28.432	46.315	1.3481	2.2047	0.220
TEEIW	-3.3405	72.293	0.5089	3.3271	0.551
BIW	-7.7712	0.475	193.005	4.4237	0.501
KUIW	-6.6127	2.361	299.538	1.9072	0.335
EGIW	-5.7279	112.996	0.4945	3.7072	0.529
WeIW	-19.626	18.688	0.329	1.1978	0.088
GoIW	-28.062	0.839	39.137	1.9627	0.296
MOIW	15.868	49.286	-------	0.0033	1.953
IW	46.782	-------	------	0.264	1.034

Source: Prepared by researchers based on statistical programs.



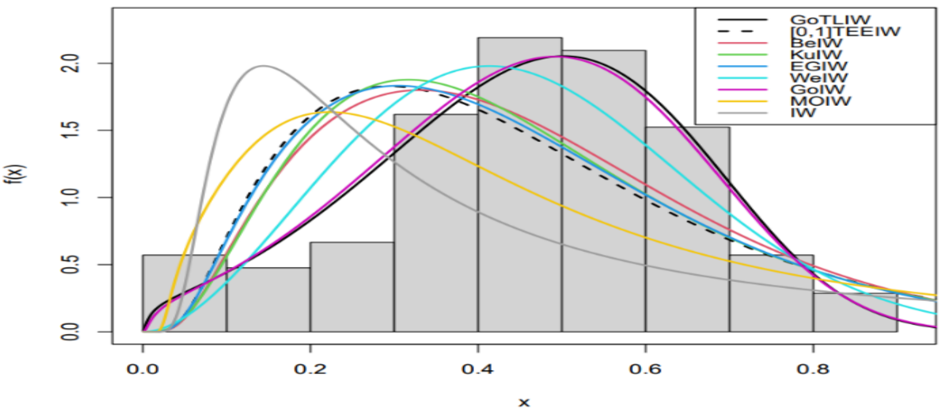


Fig. 2: fitted densities Fig. 1: fitted densities of eight distribution

Source: Prepared by researchers based on statistical programs

7. **Concluding Remarks:** The Gompertz Topp Leone Inverse Weibull GoTLIW distribution was successfully generated in this work, and its many statistical characteristics were examined. The model's shape might be ascending, falling, or unimodal. It is suggested to use the greatest likelihood method to estimate unidentified model parameters. Failure rates can be used to simulate and explain actual occurrences such bathtubs, inverted bathtubs, and bathtub increases and decreases. The GoTLIW distribution is discovered to be an advancement and a superior choice than other distributions when used with actual data sets..

References

1. Abdullah, Z. M., Hussain, N. K., Fawzi, F. A., Abdal-Hammed, M. K., & Khaleel, M. A., (2022), Estimating parameters of Marshall Olkin Topp Leon exponential distribution via grey wolf optimization and conjugate gradient with application. International Journal of Nonlinear Analysis and Applications, 13(1), 3491-3503.

2. Abid, S. H., & Abdulrazak, R. K., (2017), truncated Fréchet-gamma and inverted gamma distributions. International Journal of Scientific World, 5(2), 151-167.
3. Afify, A. Z., Ahmed, S., & Nassar, M., (2021), A new inverse Weibull distribution: properties, classical and Bayesian estimation with applications. Kuwait Journal of Science, 48(3).
4. Ahmed, M. T., Khaleel, M. A., & Khalaf, E. K., (2020), The new distribution (Topp Leone Marshall Olkin-Weibull) properties with an application. Periodicals of Engineering and Natural Sciences, 8(2), 684-692.
5. Ahmed, M. T., Khaleel, M. A., Oguntunde, P. E., & Abdel-Hammed, M. K., (2021), March). A new version of the exponentiated Burr X distribution. In Journal of Physics: Conference Series (Vol. 1818, No. 1, p. 012116). IOP Publishing.
6. Alizadeh, M., Cordeiro, G. M., Pinho, L. G. B., & Ghosh, I., (2017), The Gompertz-G family of distributions. Journal of statistical theory and practice, 11, 179-207.
7. Al-Noor, N. H., & Khaleel, M. A., (2021, March), Marshal Olkin Marshal Olkin Gompertz distribution. In AIP Conference Proceedings (Vol. 2334, No. 1, p. 090001). AIP Publishing LLC.
8. Al-Noor, N. H., Khaleel, M. A., & Assi, N. K., (2022), The Rayleigh Gompertz distribution: Theory and real applications. International Journal of Nonlinear Analysis and Applications, 13(1), 3505-3516.
9. Al-Noor, N. H., Khaleel, M. A., & Mohammed, G. J., (2021, September), Theory and applications of Marshall Olkin Marshall Olkin Weibull distribution. In Journal of Physics: Conference Series (Vol. 1999, No. 1, p. 012101). IOP Publishing.
10. Bera, W. T., (2015), The Kumaraswamy inverse Weibull Poisson distribution with applications. Indiana University of Pennsylvania.
11. Bourguignon, M., Silva, R. B., & Cordeiro, G. M., (2014), The Weibull-G family of probability distributions. Journal of data science, 12(1), 53-68.
12. Chipepa, F., & Oluyede, B., (2021), The Marshall-Olkin-Gompertz-G family of distributions: properties and applications. J. Nonlinear Sci. Appl, 14(4), 257-260.
13. Eugene, N., Lee, C., & Famoye, F., (2002), Beta-normal distribution and its applications. Communications in Statistics-Theory and methods, 31(4), 497-512.
14. Hassan, A. S., Almetwally, E. M., Khaleel, M. A., & Nagy, H. F., (2021), Weighted power Lomax distribution and its length biased version: Properties and estimation based on censored samples. Pakistan Journal of Statistics and Operation Research, 343-356.
15. Ibrahim, N. A., Khaleel, M. A., Merovci, F., Kilicman, A., & Shitan, M., (2017), Weibull Burr X Distribution Properties and Application. Pakistan Journal of Statistics, 33(5).
16. Khaleel, M. A., Abdulwahab, A. M., Gaftan, A. M., & Abdel-hammed, M. K., (2022), A new [0, 1] truncated inverse Weibull Rayleigh distribution properties with application to COVID-19. International Journal of Nonlinear Analysis and Applications, 13(1), 2933-2946.
17. Khaleel, M. A., Al-Noor, N. H., & Abdal-Hameed, M. K., (2020), Marshall Olkin exponential Gompertz distribution: Properties and applications. Periodicals of Engineering and Natural Sciences, 8(1), 298-312.

18. Khaleel, M. A., Ibrahim, N. A., Shitan, M., & Merovci, F., (2016, June), Some properties of Gamma Burr type X distribution with application. In AIP Conference proceedings (Vol. 1739, No. 1, p. 020087). AIP Publishing LLC.
19. Khaleel, M. A., Ibrahim, N. A., Shitan, M., Merovci, F., & Rehman, E., (2017), Beta burr type x with application to rainfall data. Malaysian Journal of Mathematical Sciences, 11, 73-86.
20. Khaleel, M. A., Oguntunde, P. E., Ahmed, M. T., Ibrahim, N. A., & Loh, Y. F., (2020), The Gompertz flexible Weibull distribution and its applications. Malaysian Journal of Mathematical Sciences, 14(1), 169-190.
21. Khaleel, M. A., Oguntunde, P. E., Al Abbasi, J. N., Ibrahim, N. A., & AbuJarad, M. H. (2020). The Marshall-Olkin Topp Leone-G family of distributions: A family for generalizing probability models. Scientific African, 8, e00470.
22. Khaleel, M. A., Oguntunde, P. E., Al Abbasi, J. N., Ibrahim, N. A., & AbuJarad, M. H., (2020), The Marshall-Olkin Topp Leone-G family of distributions: A family for generalizing probability models. Scientific African, 8, e00470.
23. Khan, M. S., & King, R., (2012), Modified inverse Weibull distribution. Journal of statistics applications & Probability, 1(2), 115.
24. Khan, M. S., Pasha, G. R., & Pasha, A. H., (2008), Theoretical analysis of inverse Weibull distribution. WSEAS Transactions on Mathematics, 7(2), 30-38.
25. Kundu, D., & Howlader, H., (2010), Bayesian inference and prediction of the inverse Weibull distribution for Type−II censored data. Computational Statistics & Data Analysis, 54(6), 1547-1558.
26. Maxwell, O., Chukwu, A. U., Oyamakin, O. S., & Khaleel, M. A., (2019), The Marshall-Olkin inverse Lomax distribution (MO-ILD) with application on cancer stem cell. Journal of Advances in Mathematics and Computer Science, 33(4), 1-12.
27. Merovci, F., Khaleel, M. A., Ibrahim, N. A., & Shitan, M., (2016), The beta Burr type X distribution properties with application. Springer Plus, 5, 1-18.
28. Mudholkar, G. S., & Kollia, G. D., (1994), Generalized Weibull family: a structural analysis. Communications in statistics-theory and methods, 23(4), 1149-1171.
29. Muhammed, H. Z., & Almetwally, E. M., (2020), Bayesian and non-Bayesian estimation for the bivariate inverse Weibull distribution under progressive type-II censoring. Annals of Data Science, 1-32.
30. Oguntunde, P. E., Khaleel, M. A., Adejumo, A. O., & Okagbue, H. I., (2018), A study of an extension of the exponential distribution using logistic-x family of distributions. International Journal of Engineering and Technology, 7(4), 5467-5471.
31. Oguntunde, P. E., Khaleel, M. A., Ahmed, M. T., Adejumo, A. O., & Odetunmibi, O. A., (2017), A new generalization of the Lomax distribution with increasing, decreasing, and constant failure rate. Modelling and Simulation in Engineering, 2017.
32. Rannona, K., Oluyede, B., & Chamunorwa, S., (2022), The Gompertz-Topp Leone-G Family of Distributions with Applications. Journal of Probability and Statistical Science, 20(1), 108-126.

Doi: www.doi.org/10.25130/tjaes.19.61.2.22

33. Yousof, H. M., Alizadeh, M., Jahan Shahi, S. M. A., Ghosh, T. G. R. I., & Hamedani, G. G. (2017). The transmuted Topp-Leone G family of distributions: theory, characterizations and applications. Journal of Data Science, 15(4), 723-740.